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The Taylor internal structure of weak shock waves

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G. I. Taylor's solution in 1910 for the interior structure of a weak shock wave is, with appropriate generalization, an essential component of weak-shock theory. The Taylor balance between nonlinear convection and thermoviscous diffusion is, however, endangered when other linear mechanisms – such as density stratification, geometrical spreading effects, tube wall attenuation and dispersion, etc. – are included. The ways in which some of these linear mechanisms cause the Taylor shock structure to break down when a weak shock has propagated over a large (and in some cases quite moderate) distance will be studied. Different forms of breakdown of the Taylor shock structure will be identified, both for quadratic (gasdynamic) nonlinearity and also for cubic nonlinearity appropriate to transverse waves in solid media or electromagnetic waves in nonlinear dielectrics. From this a description will be given of the fate of a nonlinear wave containing a pattern of weak shock waves, as it propagates over large ranges under the influence of linear and nonlinear mechanisms.

1. Introduction

G. I. Taylor turned his attention early to issues of significance in fluid mechanics. His second scientific paper (the first, in 1909, was on 'Interference fringes in feeble light') was published by the Royal Society in 1910 under the title 'The conditions necessary for discontinuous motion in gases'. In this brief paper, Taylor not only showed simply that gasdynamic compressions were needed to produce near-discontinuities, and that molecular diffusivity was then sufficient to sustain a near-discontinuity of permanent form, but he gave the hyperbolic tangent description of the interior of a weak shock wave that, with numerous generalizations of detail but not of spirit, is now well known as the Taylor shock structure function. The essence of that structure is a localized steady-state balance between nonlinear convection and thermoviscous diffusion. Outside the thin shock regions the balance is upset. A time-dependent conflict takes place between these nonlinear steepening and linear 'easing' mechanisms (involving also other mechanisms, such as wave-front area variation, which are generally absent within the shocks) – a conflict analysed and described in detail in the classic paper 'Viscosity effects in sound waves of finite amplitude' contributed by Sir James Lighthill to the *G. I. Taylor 70th Anniversary Volume* (Batchelor & Davies 1956).

Lighthill's paper may be regarded as initiating the research field of Nonlinear Acoustics. Among the many seminal ideas of this paper are (1) the derivation of the model equation known as Burgers' equation through a rational approximation scheme (though, characteristically, Lighthill chose not to present this in formal notation), this equation having as its travelling wave solution the Taylor shock of tanh form, (2) a proof from the Hopf–Cole exact linearization of Burgers' equation that the Taylor structure emerges as the discontinuity-bridging function in flows with strong temporal variation outside the shocks, and (3) a sketch of the extent to which the Taylor structure would still describe the shocks of a flow with strong spatial variation (due, for example, to wave-front spreading or contraction, or to density stratification in the ambient medium). Lighthill's suggestion was that those variations could be locally accommodated by suitable variation of the amplitude and width parameters of the plane-shock solution, a suggestion that we can confirm here, at any rate for moderate ranges of propagation.

Our principal aim, however, is a study of the behaviour of a weakly nonlinear wave pattern, containing weak shock waves, after propagation over very large ranges. Here the Taylor structure, valid for moderate ranges, loses its relevance in one of several possible ways.

(i) The shock may become rather thick, comparable in thickness with the overall wave scale.

(ii) It may become much thinner than a steady-state Taylor shock of the same strength.

(iii) It may be displaced by diffusive effects far from the location assigned to it by 'weak-shock theory' (Whitham 1974, p. 31).

(iv) The Taylor solution, regarded as the first term of an asymptotic expansion, may lose its dominance over the second and higher terms.

(v) The lossless nonlinear solution outside the shocks may become invalid, thereby invalidating the shock description.

(vi) The Taylor-shock solution may itself develop an internal singularity at some finite range.

These six ways in which the weak-shock-theory picture (of a weakly nonlinear dissipative wave as composed of nonlinear non-dissipative 'simple-wave' regions separated by thin shock waves, with Taylor structure, located at those positions for a discontinuity which would conserve mass and momentum) can break down have been identified in a number of recent studies (Crighton & Scott 1979; Scott 1981*b*; Nimmo & Crighton 1986; Lee-Bapty & Crighton 1986*a*, *b*). In some problems several such non-uniformities arise at the same typical range, while in others they occur sequentially if at all. Other sources of large-range non-uniformity may of course arise in flows with dissipative mechanisms other than distributed thermoviscosity (by which we mean dissipation associated with molecular diffusion of heat and momentum, characterized by coefficients of thermal conductivity and viscosity). For example, dissipation associated with a relaxing degree of freedom (principally the vibrational mode of nitrogen in atmospheric air) is often dominant in unbounded flows, while attenuation and dispersion in Stokes layers adjacent to tube walls is often dominant in confined flows, while there are many other non-dissipative linear mechanisms which may lead to spatial variation. In this paper we are concerned solely with thermoviscous dissipation; some treatment of the corresponding problems for a relaxing gas is given in Crighton & Scott (1979).

With regard to the first possibility (i) mentioned, it is clear that the near-discontinuity description no longer holds, that a 'global non-uniformity' has arisen, and that a new description of the whole wave is called for. On the second, a region thinner than the steady Taylor width for a given shock strength must be governed by linear diffusion rather than the Taylor–Lighthill balance – as first apparently noted by Naugol'nykh (1973). An error function replaces Taylor's tanh, with a different width law and the possibility of a global non-uniformity later on. For the third, Lighthill (1956, equations (159) and (212)) shows that for plane flow viscous effects can cause the 'centre' of a Taylor shock to drift through the waveform far from the weak-shock-theory location (which generally displaces the shock too far); and the effect is more pronounced for cylindrical and spherical waves. However, we have not yet found a case in which this 'translational non-uniformity' is not accompanied by another non-uniformity, local or global. The fourth source of non-uniformity of weak-shock theory was first identified by Crighton & Scott (1979). If the flow outside the shock varies too rapidly, then the Taylor–Lighthill balance can no longer be maintained. Then, if this is the only non-uniformity at the range considered, the Taylor shock gives way to an *evolutionary* (non-steady-state) shock, this in turn generally giving way at longer ranges to a linear error-function shock, and possibly to a global non-uniformity at still greater ranges. When the fifth type of non-uniformity arises, the shocks again turn out to be linear error functions, but outside them the flow is no longer nonlinear, changing instead only because of linear geometrical constraints. The sixth possibility applies only to problems involving cubic or higher nonlinearity, such as, for example, those involving shear or torsional waves in solids, or electromagnetic waves in nonlinear dielectrics. There is, for such problems, also a Taylor shock structure – but now an internal singularity can develop at finite range, together with indeterminacy of the flow outside the shocks, unless a certain relation is maintained between the signals on either side of the shock (Lee-Bapty & Crighton 1986*a*). This leads to completely new features, unavailable to gasdynamics, such as the refraction of characteristics passing through a shock, and slow algebraic matching of the shock to the flow on one side with rapid exponential matching on the other. These features would all be missed if it were tacitly assumed that a smooth Taylor-like transition could always be found between two arbitrary signal levels; it could for a gasdynamic shock, but not generally for higher-order nonlinearity.

We shall not go into details here on the specific functional forms that reveal the various non-uniformities. These details are given in the papers cited and in further papers dealing with the propagation of single hump disturbances and with the propagation of cylindrical N-waves through a stratified atmosphere (the sonic-boom problem). The aim here is rather to present some overview and perspective – essential in nonlinear problems, where generality is not achieved by superposition and where it is not clear whether results for a particular initial condition or a particular linear mechanism exemplify broad classes of generally similar behaviour.

Section 2 describes briefly the genesis of the model equations – generalized or modified Burgers' equations – whose (local) travelling waves are Taylor shocks. Exact solutions are known only for the ordinary plane Burgers' equation, and we therefore attack the others with asymptotic techniques backed up by numerical work by others and ourselves. Section 2 therefore gives appropriate dimensionless forms and small parameters for the problems most completely solved to date, involving comparable weak nonlinearity, thermoviscous diffusion and geometrical area variation of the wave front. Section 3 then gives the classification of results for two particular initial wave forms, the N-wave and the sinusoid, for all area variations. Two aspects are of particular interest. The first concerns the types of non-uniformity that arise in the 'lossless waves separated by thin Taylor shocks' description and of the new balances that take over following a non-uniformity. The second is the matter of the ultimate behaviour of the wave – whether weak-shock theory is valid to indefinitely large ranges, or whether, at the other extreme, the wave subsides into 'old-age' decay governed by purely linear mechanisms. Section 4 discusses briefly the evolution of waves under cubic and higher modifications of Burgers' equation, and §5 gives some credence to the asymptotic structure proposed by making qualitative and quantitative comparisons with published numerical calculations. Some of the myriad ways in which this programme of asymptotic analysis of model equations of nonlinear acoustics – essentially originated by Taylor and Lighthill – could be extended are mentioned in §6.

2. Model equations

We consider disturbances to a thermoviscous fluid whose ambient state is one of homogeneous equilibrium with constant values of the thermodynamic variables in standard notation. The disturbances are assumed to propagate, with negligible reflection, in the direction of increasing x, where x is a space coordinate along a physical horn or ray tube, and all fluctuations are assumed uniform over a section of the tube or horn. A motion of velocity amplitude U_0 at station $x = x_0$ generates the motion in the fluid $x \geqslant x_0$, ω^{-1} is the typical timescale of the imposed motion, $k_0^{-1} = a_0 \omega^{-1}$ is the typical lengthscale of a linear acoustic motion. Nonlinear acoustics deals with motions for which the parameters U_0/a_0, $k_0 \delta/U_0$ and $(k_0 L)^{-1}$ are small and comparable and there is interest in propagation ranges of order $(a_0/U_0) k_0^{-1}$ or larger and spatial gradients of order $(U_0/k_0\delta)$ times the maximum gradient in a purely linear wave. Here U_0/a_0 is a source Mach number; $k_0\delta/U_0$ is an inverse Reynolds number based on U_0, the wavelength k_0^{-1}, and that combination δ of the diffusivities (the 'diffusivity of sound' – Lighthill 1956) relevant to acoustic wave damping; and $L = |\mathrm{d}/\mathrm{d}x \ln A(x)|^{-1}$ is a length characterizing the rate of change of ray-tube or wave-front area $A(x)$ (in most cases L may be identified with x_0).

A multiple scales expansion of the gasdynamic equations, with the usual suppression of secular terms, then readily leads, on reversion to physical variables, to the equation

$$\frac{\partial u}{\partial t} + (a_0 + \tfrac{1}{2}(\gamma+1)\, u)\frac{\partial u}{\partial x} + \tfrac{1}{2} a_0\, u \frac{\mathrm{d}}{\mathrm{d}x} \ln A(x) = \frac{\delta}{2}\frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

for the velocity fluctuation. This holds uniformly, as the small parameters vanish independently, to times $O(\eta^{-1}\omega^{-1})$, where η is the smallest of the three small parameters; see Lighthill (1956), Leibovich & Seebass (1974), Rudenko & Soluyan (1977) and Crighton (1986) for derivations at varying levels of formality.

For plane flow (2.1) is the ordinary Burgers' equation which is linearized to the diffusion equation

$$\frac{\partial \psi}{\partial t} = \tfrac{1}{2}\delta \frac{\partial^2 \psi}{\partial X^2} \tag{2.2}$$

by the Hopf–Cole transformation (Bäcklund transformation)

$$u = -\delta\left(\frac{2}{\gamma+1}\right)\frac{\partial}{\partial X}\ln \psi, \quad X = x - a_0 t. \tag{2.3}$$

Lighthill (1956), Whitham (1974) and Rudenko & Soluyan (1977) give extensive discussion of features of the nonlinear plane wave evolution revealed by the Hopf–Cole transformation. For the generalized Burgers' equations (GBE's) represented by (2.1) with $A(x) \neq$ constant it has been proved (Nimmo & Crighton 1982) that there is no Bäcklund transformation mapping the solutions of (2.1) onto solutions of the same equation, of the corresponding linear equation, or onto any other GBE of the class (2.1). Asymptotic and numerical methods are therefore the only tools currently available.

The linear terms on the left of (2.1) express the fact that in the geometrical acoustics high-frequency limit the quantity $A^{\frac{1}{2}}(x)\, u(x, t)$ is constant on the linear characteristics $\mathrm{d}x/\mathrm{d}t = a_0$. For the important cases of cylindrical and spherical waves, $A = (x_0/x)$ and $(x_0/x)^2$ respectively, and the generalizing linear term is $ja_0u/2x$ with $j = 1, 2$ respectively. Various transformations of (2.1) are useful at particular stages of the motion, of which the most important throws the whole of the left side into plane-wave form (and thereby allows the lossless equation, with $\delta = 0$, to be integrated by characteristics) at the expense of introducing an effective viscosity which depends strongly on time, or range from x_0. Define dimensionless variables according to

$$\left.\begin{aligned} u = U_0 f(\omega t) \quad \text{at } x = x_0, \quad \theta = \omega\left(t - \frac{x - x_0}{a_0}\right), \\ \left(\frac{A}{A_0}\right)^{\frac{1}{2}} u = U_0 q, \quad Z = \tfrac{1}{2}(\gamma+1)\frac{U_0}{a_0}k_0 \int_{x_0}^{x} \left(\frac{A_0}{A}\right)^{\frac{1}{2}} \mathrm{d}x, \\ G(Z) = \left(\frac{A}{A_0}\right)^{\frac{1}{2}}(Z), \quad \epsilon = \frac{1}{(\gamma+1)}\frac{k_0\delta}{U_0}, \end{aligned}\right\} \tag{2.4}$$

and then the wave evolution is governed by

$$\frac{\partial q}{\partial Z} - q\frac{\partial q}{\partial \theta} = \epsilon G(Z)\frac{\partial^2 q}{\partial \theta^2}, \tag{2.5}$$

with $q(\theta, 0) = f(\theta)$.

For cylindrical waves

$$Z = \tfrac{1}{2}(\gamma+1)\left(\frac{U_0}{a_0}\right)(k_0 x_0)\, 2\left[\left(\frac{x}{x_0}\right)^{\frac{1}{2}} - 1\right], \quad G(Z) = 1 + \frac{4}{(\gamma+1)(U_0/a_0)(k_0 x_0)} Z, \tag{2.6}$$

and for spherical waves

$$Z = \tfrac{1}{2}(\gamma+1)\left(\frac{U_0}{a_0}\right)(k_0 x_0)\ln\left(\frac{x}{x_0}\right), \quad G(Z) = \exp\left[\frac{2}{(\gamma+1)(U_0/a_0)(k_0 x_0)} Z\right]. \tag{2.7}$$

The GBE (2.5) does not follow exactly from (2.1), but has precisely the same formal validity as (2.1). In (2.1) the nonlinear, geometrical and diffusive terms are all assumed to be locally small, and formally equivalent expressions for them may be obtained using the local balance $\partial u/\partial t \approx -a_0\, \partial u/\partial x$. For further details see Crighton (1979, 1986).

Our interest is in the evolution over arbitrarily large 'ranges' Z (or the full range of Z corresponding to $x_0 \leqslant x < \infty$) in the fully nonlinear limit $\epsilon \to 0$. As noted earlier, however, the problem contains three small parameters, of which one is ϵ and a second, U_0/a_0 say, has been used to define the slow space variable Z. The third is the product of a small Mach number U_0/a_0 and a large Helmholtz number $k_0 x_0$, $Z_0 = \frac{1}{2}(\gamma+1)\,(U_0/a_0)\,(k_0 x_0)$. This may be taken as fixed, corresponding to letting the diffusivities decrease with other parameters held fixed, and would be appropriate to, say, sonic-boom propagation in the atmosphere (provided density variations were taken into account also). Applications in underwater acoustics often correspond, however, to considerable variations of the drive level U_0 with fixed values of the other quantities, and in that case the limit $\epsilon \to 0$ must be taken with ϵZ_0 fixed, $\epsilon Z_0 = \alpha x_0$ being the product of the small-signal attenuation coefficient $\alpha = \delta\omega^2/2a_0^3$ with x_0.

The travelling-wave solution to (2.5) in the plane-wave case $G(Z) = G_0$, constant, is

$$q = V + \tfrac{1}{2}(V_+ - V_-)\tanh\left[\frac{(V_+ - V_-)(\theta + VZ)}{4\epsilon G_0}\right], \tag{2.8}$$

increasing from V_- at $\theta = -\infty$ to V_+ at $\theta = +\infty$, travelling at speed $\mathrm{d}\theta/\mathrm{d}Z = -V = -\frac{1}{2}(V_+ + V_-)$, and of scale $\epsilon G_0/(V_+ - V_-)$. This is G. I. Taylor's (1910) solution for the structure of a weak thermoviscous shock. For $G(Z)$ not constant there is no travelling wave of permanent form, but (2.8) still holds as the leading term of an asymptotic expansion in regions of thickness $|\theta-\theta_s(Z)| = O(\epsilon)$ around particular phase locations $\theta_s(Z)$ corresponding to the discontinuities of weak-shock theory; and then $V_\pm$ and G_0 have their *local* values $V_\pm(Z)$ and $G(Z)$.

In areas of mechanics other than gasdynamics the coefficient of the quadratic term $u\,\partial u/\partial x$ in (2.1) may vanish identically because of symmetry requirements, as in the case of shear or torsional waves in isotropic elastic solids or electromagnetic waves in nonlinear dielectrics (see Nariboli & Lin 1973; Sugimoto, Yamane & Kakutani 1982; Gorschkov, Ostrovsky & Pelinovsky 1974; Lee-Bapty & Crighton 1986*a*). There are also hydrocarbon and fluorocarbon fluids for which this coefficient vanishes along a curve in a thermodynamic space, in the neighbourhood of which expansion and compression shocks are both possible. References are given by Cramer & Kluwick (1984). In such cases, a modified (cubic) Burgers' equation replaces (2.1) provided the dissipative mechanism is diffusive and dispersion negligible, so that a solid medium must be viscoelastic, while atomic resonances must be avoided in the nonlinear optics case to minimize dispersion. Detailed derivations are given in the references cited. The dimensionless equation, in variables corresponding closely to those in (2.5), is

$$\frac{\partial q}{\partial Z}+q^2\frac{\partial q}{\partial \theta} = \epsilon\frac{\partial^2 q}{\partial \theta^2} \tag{2.9}$$

for plane motion. The Taylor shock solution is

$$q = f(\xi), \quad \xi = \theta-\theta_0-VZ,$$

where

$$\frac{\xi}{3\epsilon} = \int\frac{\mathrm{d}f}{(f^3-3Vf+A)} \tag{2.10}$$

and the integration is between adjacent zeros of the cubic. If the zeros, f_1, f_2 and $-(f_1+f_2)$, are distinct then f approaches one of them exponentially as $\xi\rightarrow+\infty$ and an adjacent one exponentially as $\xi\rightarrow-\infty$. If $f_1 = f_2$ then the approach to f_1 is only algebraic. If $f\rightarrow 0$ as $\xi\rightarrow+\infty$ then f can be found in simple form,

$$f = \frac{a\exp\left(-\dfrac{a^2\xi}{3\epsilon}\right)}{\left[1+\exp\left(-\dfrac{2a^2\xi}{3\epsilon}\right)\right]^{\frac{1}{2}}}, \quad a = (3V)^{\frac{1}{2}}. \tag{2.11}$$

In this case there is a non-singular transition to an arbitrary specified level as $\xi\rightarrow-\infty$, and such a 'head shock' poses no difficulty when incorporated locally into a description in which the signal a behind the shock is a function of Z. The difficulty arises in the more general case represented by (2.10). When this is used locally to resolve the fine structure of a discontinuity, the zeros f_1, f_2 and $-(f_1+f_2)$ are functions of Z determined by the initial data and the solutions of (2.9) with $\epsilon = 0$, which holds outside the shocks. Then, even though initially $f_1(Z)$ and $f_2(Z)$ may be adjacent zeros, with an acceptable Taylor shock linking them, it turns out that the third zero coincides with $f_2(Z)$, say, at some finite range Z_1, and would for $Z > Z_1$ move between f_1 and f_2 and lead to a non-integrable singularity within the shock itself. How this must be avoided will be explained shortly. It is clearly an important feature of the nonlinear wave mechanics of dissipative systems with nonlinearity of cubic or higher order, and seems not to have been addressed except in the recent work of Lee-Bapty & Crighton (1986*a*). Standard discussions of shock fitting for general nonlinearity

$Q'(q)\,\partial q/\partial\theta$, say, (cf. Whitham 1974, pp. 33, 34), assume that conditions outside the shock can be determined in the way so familiar in gasdynamics – from the initial data via characteristics – and will pose no threat of singularity for the shock itself. For cubic $Q(q)$ it can be shown explicitly that the situation is in general quite different. Lee-Bapty & Crighton (1986*a*) show that coalescence of zeros for a unit N-wave takes place at $Z_1 = 10$, and at a range $Z_1 = 9.601$ for a unit sinusoid as initial data. Section 4 below describes how the dilemma for $Z > Z_1$ is resolved.

Anisotropy of the medium is a common feature and leads to interesting model equations and wave structure. Consider, for example, an isothermal atmosphere with exponential variation of the mean density with height z and scale height H. Then if r is the radial coordinate and $z = r\cos\theta$, the model equation generalizing (2.1) for cylindrical or spherical waves to take account of density stratification is

$$\frac{\partial u}{\partial t}+(a_0+\tfrac{1}{2}(\gamma+1)\,u)\frac{\partial u}{\partial r}+\frac{ja_0\,u}{2r}-\frac{a_0\,u}{2H}\cos\theta=\tfrac{1}{2}\delta_0\exp\left(\frac{r}{H}\cos\theta\right)\frac{\partial^2 u}{\partial r^2}, \qquad (2.12)$$

where δ_0 is the diffusivity at $z = 0$. The evolution takes place along rays labelled by θ, and it is quite possible to have, at given range r, shock-free flows for one range of θ, flows with fully developed Taylor shocks for another range of θ, and flows decaying in linear old age for yet another range of θ. Spherical wave evolution, $j = 2$ in (2.12), was discussed in Lee-Bapty & Crighton (1986*b*) with sinusoidal wave propagation in the ocean in mind. Cylindrical wave propagation, with $j = 1$ and an N-wave at $r = r_0$, models the sonic-boom problem in the atmosphere, and will be discussed elsewhere.

The remainder of this paper discusses only isotropic media, where the generalized and modified Burgers' equations (2.5) and (2.9) are appropriate.

3. Wave-front area variations

We outline here a typical matched-expansion approach to the solution of (2.5) as $\epsilon\to 0$ (with $Z_0 = O(1)$ or $\epsilon Z_0 = O(1)$), stating the types of non-uniformity that can arise in the weak-shock-theory description valid for moderate ranges, sketching the evolution following different types of non-uniformity and, following Nimmo & Crighton (1986), giving a classification of various scenarios and routes to old-age decay for the two important initial signals of N-wave and sinusoidal form.

3.1. *The lossless expansion*

For $O(1)$ values of θ, Z, assume

$$q \sim q_0(\theta,Z)+\epsilon q_1(\theta,Z)+\ldots. \qquad (3.1)$$

Then q_0 is a lossless simple wave, satisfying

$$\frac{\partial q_0}{\partial Z}-q_0\frac{\partial q_0}{\partial\theta}=0, \quad q_0(\theta,0)=f(\theta), \qquad (3.2)$$

with implicit characteristic solution

$$q_0=f(\phi), \quad \phi=\theta+Zf(\phi).$$

Corrections $q_1, q_2, \ldots$ can also be found, and show in some cases that $\epsilon q_2 \sim q_1$ for large Z (see later). The lossless expansion is invalid in the embryo-shock region, in which a triple-valued q_0 is about to be produced and diffusive effects are important, and invalid in the thin shock regions which provide a rapid transition from one branch to another of the multi-valued wave form described by q_0 for $Z > Z_1$ (shock-formation range). In the embryo-shock region the shocks have not yet developed their steady-state Taylor form and the leading term of an expansion in 'embryo-shock variables',

$$\hat{\theta} = \frac{\theta-\theta_1}{\epsilon^{\frac{3}{4}}}, \quad \hat{Z} = \frac{Z-Z_1}{\epsilon^{\frac{1}{2}}}, \quad q \sim \epsilon^{\frac{1}{4}}\hat{q}_0(\hat{\theta},\hat{Z})+\ldots, \tag{3.3}$$

satisfies

$$\frac{\partial \hat{q}_0}{\partial \hat{Z}} - \hat{q}_0\frac{\partial \hat{q}_0}{\partial \hat{\theta}} = G(Z_1)\frac{\partial^2 \hat{q}_0}{\partial \hat{\theta}^2} \tag{3.4}$$

if infinite gradient in q_0 is first produced at range Z_1 and phase θ_1. Equation (3.4) is the plane Burgers' equation, and a solution can be found which matches q_0 for $Z < Z_1$ and for $Z > Z_1$ outside the shocks, and matches (3.7) below within the shocks (see Lighthill 1956, equation (171), and Crighton & Scott 1979, equation (4.58)). The solution in this region is the key to some important later details. Outside the shock regions, the lossless q_0 generally continues to be valid well beyond shock formation, and between the shocks describes a series of straight-line ramps with slope $-Z^{-1}$.

3.2. *Steady-state Taylor shocks*

Assume that the triple-valued solution q_0 (for $Z > Z_1$) is made single-valued by a region of rapid change – a shock – around $\theta = \theta_s(Z)$, say, where $\theta_s(Z)$ is to be determined by matching. The shock thickness turns out to be $O(\epsilon)$, so that $\theta^* = (\theta-\theta_s)/\epsilon$ and Z are suitable 'shock variables', with a shock expansion

$$q(\theta^*, Z, \epsilon) \sim q_0^*(\theta^*, Z) + \epsilon q_1^*(\theta^*, Z)+\ldots, \tag{3.5}$$

with matching of q_0^* as $\theta^* \to \pm\infty$ to the limiting values $q_\pm$ of q_0 as $\theta \to \theta_s \pm$. The function $(q_0^* + \theta_s')$ satisfies

$$-u\frac{\partial u}{\partial \theta^*} = G(Z)\frac{\partial^2 u}{\partial \theta^{*2}}, \tag{3.6}$$

reflecting a local Taylor–Lighthill balance, with geometrical area variations appearing only parametrically in $G(Z)$. Solving for u as $h(Z)\tanh\{h(Z)[\theta^*-\theta_0^*(Z)]/2G(Z)\}$, where h and θ_0^* are functions of integration, and performing the matching leads to

$$q_0^* = V(Z) + \tfrac{1}{2}(q_+ - q_-)\tanh\left\{\frac{(q_+-q_-)[\theta + \int V(Z)\,dZ - \epsilon\theta_0^*(Z)]}{4\epsilon G(Z)}\right\}, \tag{3.7}$$

which is locally of precisely the Taylor form (2.8), with a shock propagation velocity $\theta_s'(Z) = -V(Z) = -\frac{1}{2}(q_+ + q_-)$ which is the discontinuity velocity of weak-shock theory.

Observe, however, the following. The shock width is of order $\epsilon G(Z)/(q_+ - q_-)$ and may not be uniformly $O(\epsilon)$ for large Z. Second, q_0^* is antisymmetric about a centre at $\theta_s(Z)+\epsilon\theta_0^*(Z)$ in which $\theta_s(Z)$ is the weak-shock-theory discontinuity location and $\epsilon\theta_0^*(Z)$ is Lighthill's 'shock displacement due to diffusivity'. The function $\theta_0^*(Z)$ needs second-order matching, of $q_0+\epsilon q_1$ to $q_0^*+\epsilon q_1^*$, for its determination in general, though sometimes it can be efficiently calculated from some integral conservation principle, while in other cases symmetry conditions show that $\theta_0^*(Z)$ must have a fixed value. When $\theta_0^*(Z)$ is not constant it is frequently found that $\epsilon\theta_0^*(Z) \gg \theta_s(Z)$ as $Z \to \infty$; for N-waves, $\theta_s(Z) \sim Z^{\frac{1}{2}}$, while $\theta_0^*(Z) \sim Z^{\frac{1}{2}}\ln Z$, $Z^{\frac{3}{2}}$ and $Z^{-\frac{1}{2}}e^Z$ for plane, cylindrical and spherical waves respectively. Shock displacement due to diffusivity is therefore a possible source of non-uniformity at large Z. More important, however, is the non-uniformity revealed by the correction ϵq_1^* to Taylor's q_0^*. A general expression for q_1^* can be found, but is too lengthy to write down here. The essential point is that it involves not only the quantities $q_{\pm}(Z)$ just outside the shock at range Z and the function $G(Z)$ but also their *derivatives*, and therefore the idea that the Taylor–Lighthill balance of (3.6) must be upset if wave-front area variations cause too rapid a change just outside the shock can be quantified by looking at the ratio $\epsilon q_1^*/q_0^*$ for large Z. When this ratio becomes $O(1)$, Taylor's description of the shock interior certainly fails, though what replaces it depends on whether this is the first non-uniformity to arise, or whether another arises at the same typical large Z.

3.3. *Weak-shock theory*

At this point it may be convenient to state exactly what is meant by this term. In weak-shock theory, one solves the lossless equation (3.2) with jump discontinuities (only compressions are ever needed – Taylor 1910) within intervals of phase θ where q_0 is triple-valued. The discontinuities are inserted in accordance with the 'equal-areas rule' with respect to either the initial wave profile $f(\theta)$ or the evolved profile $q(\theta, Z)$ (Lighthill 1956; Whitham 1974) and the analytical expression of this is the formula $\theta_s'Z) = -\frac{1}{2}(q_+ + q_-)$, a differential equation for the shock path. It is tacitly assumed that a smooth transition could be found to replace the discontinuity if a suitable dissipative mechanism were introduced, and that if diffusivity is that mechanism then the transition is of steady Taylor form, implying a precise knowledge of the shock width for a given strength. From what has been said above it is evident that weak-shock theory must frequently fail to hold uniformly for large ranges Z, and that it can fail in a number of different ways depending on the initial wave profile and on the area function $G(Z)$.

3.4. *Non-uniformities in weak-shock theory*

Weak-shock theory is predicated upon the following assumptions:

(A) the shock thickness is small compared with the overall wave scale;

(B) the shock displacement due to diffusivity is a small fraction of the weak-shock-theory displacement;

(C) Taylor's solution is valid at leading order within the shocks;

(D) the lossless flow outside the shocks is a correct leading-order description (i.e. $\epsilon q_1 \ll q_0$, where q_0 is given by (3.2));

(E) for cubic and higher-order nonlinearity, no singularity must arise within the shock itself.

We know of no case in which (B) is violated first, though in many cases it is violated at the same typical range as (C) and we shall therefore include it within (C). Condition (E) will be set aside until §4.

The simplest case (apart from that of uniform validity of weak shock theory) arises when (A) and (C) are simultaneously violated (and then (D) is also). In this case we have a *global non-uniformity* in which fine shock structure has been lost and nothing of weak shock theory is true anywhere on the wave for $Z >$ some large $Z_*(\epsilon)$. In appropriate scaled variables $\bar{q}$, $\bar{Z}$ and for all θ it is found that following such a non-uniformity $\bar{q}_0$ satisfies the full generalized Burgers' equation

$$\frac{\partial \bar{q}_0}{\partial \bar{Z}} - \bar{q}_0 \frac{\partial \bar{q}_0}{\partial \theta} = H(\bar{Z}) \frac{\partial^2 \bar{q}_0}{\partial \theta^2}, \tag{3.8}$$

where $H(\bar{Z})$ is close to $G(Z)$. As $\bar{Z} \to 0$, $\bar{q}_0$ must match the lossless solution q_0 outside the shocks and the Taylor solution q_0^* within them. Equation (3.8) states that linear evolution under geometrical constraints, nonlinear convection and diffusion are comparable across the whole wave. With the exceptions of plane waves, and of a similarity solution for the cylindrical version of (3.8), with $H(\bar{Z}) = \bar{Z}$ (whose significance is discussed by Scott 1981*a*), no solutions at all of (3.8) are known. It can be argued, however, that as $\bar{Z} \to \infty$ the nonlinear term becomes small everywhere (this follows immediately from a formal scaling), and the wave subsides into old age under linear dynamics. The functional form of this decay can often be found; for sinusoidal $f(\theta)$, $\bar{q}_0 \sim A(Z) \sin(\theta - \bar{\theta})$ where $A'(Z) = -H(Z) A(Z)$, and the old-age decay is pinned down aside from a purely numerical amplitude factor in $A(Z)$ and the phase constant $\bar{\theta}$. Nothing short of the exact solution to (3.8) (with matching as $\bar{Z} \to 0$), by analytical or numerical means, can provide these numerical factors. Much is gained from this approach; all the scalings are determined, as are the ranges at which changes in wave character occur, and the numerical problems are reduced to solving (3.8) once, instead of the original equation for each ϵ.

Suppose now that (*C*) is violated before (*A*) or (*D*), at $Z \sim Z_*(\epsilon)$. Then the shock is no longer a Taylor shock–but it is still a shock in the sense of a narrow region outside which lossless nonlinear dynamics goes on. New scaled variables $\bar{q}$, $\bar{Z}$ *and* $\bar{\theta}$ are now needed, in terms of which, not surprisingly, a full generalized Burgers' equation (3.8) holds – but now only in a narrow region, following this *local non-uniformity*. The shock has become evolutionary, rather than steady state. The evolutionary solution $\bar{q}_0$ must again match the Taylor shocks and the lossless solution as $\bar{Z} \to 0$ for appropriate scaled values of θ – but the essential point is that it must *continue* to match the lossless solution as $\bar{Z}$ increases to large positive values, because the lossless solution is valid indefinitely into the 'future' ($\bar{Z} \to +\infty$) outside the shock. As $\bar{Z} \to +\infty$, the nonlinear term in (3.8) becomes small, and the solution $\bar{q}_0$ tends to an error-function solution of the linear version of (3.8) (this was proved in the Appendix to Crighton & Scott 1979). The amplitude of this *linear shock* is completely fixed by the matching to q_0 which, we emphasize again, holds even as $\bar{Z} \to \infty$.

In a sense, weak-shock theory has not yet been violated, though the shock structure is far from Taylor-like. But one must now ask of the linear evolutionary shock the questions earlier addressed to the Taylor shock – and one finds, typically, that the linear shock thickens, and violates condition (*A*), at some typical range $Z_{**}(\epsilon)$. This second long-range non-uniformity is global, but fortunately, involves only the linear form of (3.8), in scaled variables $\bar{\bar{q}}_0$ and $\bar{\bar{Z}}$, across the whole wave (all θ). A solution can be constructed, matching the lossless solution and the linear error-function shock (for example, by taking these as initial values for the linear form of (3.8) and letting $\bar{\bar{Z}} \to +\infty$), and in this case all details of the old-age decay, including the amplitude coefficient, can be determined without recourse to numerics. We have, in particular, determined the old-age decay of *N*-waves and sinusoidal waves completely in all cases where the local-followed-by-global non-uniformity scenario applies, these cases including freely spreading spherical waves (Crighton & Scott 1979; Scott 1981*b*; Nimmo & Crighton 1986).

Most cases are covered by these two main routes to old age. Exceptions are the following. A local non-uniformity leading to an evolutionary and, subsequently, a linear error-function shock need not be followed by a global non-uniformity. The error function can remain thin (it is in fact thinner than the Taylor shock of the same strength, and therefore is governed by linear dynamics, as suggested by Naugol'nykh 1973), but a non-uniformity may arise in the outer solution at still larger range, ϵq_1 becoming comparable with q_0. When this happens the appropriate replacement $\tilde{q}_0$ for the simple wave q_0 is found to satisfy the linear lossless equation

$$\frac{\partial \tilde{q}_0}{\partial \tilde{Z}} = 0, \tag{3.9}$$

with matching as $\tilde{Z} \to 0$ to the nonlinear simple wave q_0. Thus the variation in the outer flow is entirely associated with linear wave-front area variation. Inside the shocks we also have linear dynamics – but the overall behaviour, although governed by linear equations everywhere, is hardly of the kind that one would associate with 'old age'. Consider, for example, $f(\theta) = \sin\theta$ for the initial condition. Then at large physical ranges x for which the simple wave description still applies, q_0 is a periodic sawtooth, decreasing in amplitude as Z^{-1},

$$q_0 \sim \frac{\pi - \theta}{Z} \quad (0 < \theta < 2\pi). \tag{3.10}$$

In circumstances where a non-uniformity arises in this description, Z actually tends to a finite value Z_∞ as the physical range x tends to infinity, and therefore q_0 tends to a frozen sawtooth

$$\tilde{q}_0 \sim \frac{\pi - \theta}{Z_\infty}, \tag{3.11}$$

which satisfies (3.9), and (3.9) in turn must follow from (2.5) if $\tilde{Z} = (Z_\infty - Z)/\epsilon^\alpha$ for any $\alpha > 0$. In terms of u we thus have a sawtooth with amplitude varying as $A^{-\frac{1}{2}}(x)$, the discontinuities resolved by linear error-function shocks whose thickness remains small to arbitrary large x.

Second, the evolutionary and then linear shock may not thicken, and neither need a non-uniformity subsequently arise in the outer solution. Then the nonlinear simple wave ((3.10) for sinusoidal waves) continues to infinity, with linear shocks. Weak-shock theory has not been violated, but the shock structure is not Taylor's. The shock is thinner than Taylor's, and therefore linear. However, its fixed thickness means that linear dynamics never takes over everywhere, and the wave never becomes a linear 'sound wave'. The possibility of linear shock dynamics was first raised by Naugol'nykh (1973). He, however, had no systematic way of examining all the non-uniformity scenarios that are possible, and was unable to quantify the idea to the extent now demonstrated here, nor was he able to relate it to other sources of non-uniformity in weak-shock theory.

3.5. *Classification*

A detailed classification of the sequence of non-uniformities for geometrical area variations coupled with quadratic nonlinearity and thermoviscous diffusion is given in Nimmo & Crighton (1986). The following is a simple statement of the essentials for sinusoidal and N-wave initial conditions; one-signed single-pulse problems raise other issues not considered here.

Sinusoidal waves

(*a*) If $A(x)$ decreases at least as fast as $\exp(-\alpha x)$ for some $\alpha > 0$, then there is no non-uniformity at large x. Weak-shock theory holds to $x = \infty$ and the shocks retain Taylor structure.

(*b*) If $A(x)$ decreases less rapidly than $\exp(-\alpha x)$ for any $\alpha > 0$, or increases no more rapidly than x^λ for $\lambda < 2$, the first and only non-uniformity is of the gross or global kind. All effects contained in (2.5) become important everywhere (in θ) over ranges $Z \sim Z_*(\epsilon)$. At any asymptotically larger Z the wave becomes weak and decays into linear old age.

(*c*) If $A(x)$ diverges as x^2 or more rapidly, the first non-uniformity is localized in the shocks, which become evolutionary, and governed by (3.8) with $H(\bar{Z})$ fully variable. The shocks then assume a linear error-function form, while the lossless solution persists outside them. A shock will now thicken provided $A'/A \to 0$, and will provoke a global non-uniformity in which linear dissipative dynamics operates over the whole wave as old age sets in. If $A'/A \to \beta$, $0 < \beta \leqslant \infty$, no shock thickening takes place. Instead, a non-uniformity arises in the lossless solution. Outside the shocks (and already in them) nonlinearity is small and the wave has a frozen form in Z, the evolution of $u(x, t)$ being dictated solely by area variations. Inside the shocks we have linear diffusive decay.

Scenario (*a*) is exemplified by the horn of exponentially decreasing section, (*b*) by cylindrical waves ($\lambda = 1$). Scenario (*c*) with $A'/A \to 0$ is exemplified by spherical waves ($\lambda = 2$) and with $A'/A \to \beta$ by the exponentially diverging horn.

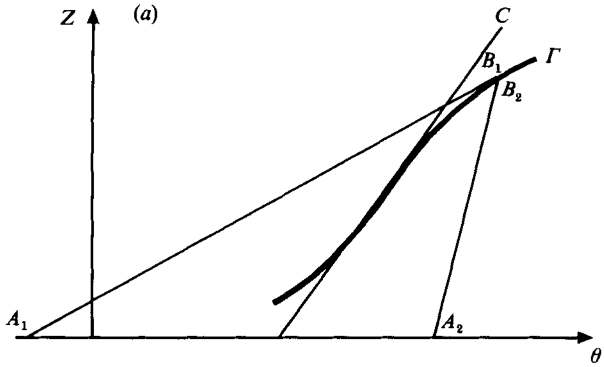
N-waves

The situation is similar, with one significant difference. Suppose that $A(x)$ increases faster than x^{λ} for any λ. Then the evolutionary shock produced by the localized non-uniformity will thicken if $A'/A \to 0$ and provoke a global non-uniformity as for sinusoidal waves. If, however, $A'/A \nrightarrow 0$, no non-uniformity is this time to be found in the lossless solution (where $q = q_0$ plus exponentially small terms, rather than the algebraic terms ϵq_1, etc.). The nonlinear lossless solution is valid indefinitely but the shocks are thin and have error-function rather than Taylor form. Naugol'nykh's shock-thickness criterion applies, but old age does *not* follow; the wave remains nonlinear outside linear shocks, and weak-shock theory remains valid.

4. Higher-order nonlinearity – Modified Burgers' equation

Here we consider the Taylor type of shock structure for cubic and higher-order (polynomial) nonlinearities coupled with diffusive dissipation. The structure is given in (2.10), in which the transition is between adjacent zeros of the cubic, if we deal specifically with cubic nonlinearity and plane waves. Discussion of (2.10) (for example, in Whitham 1974, p. 31) invariably assumes that conditions on either side of the shock have been prescribed as corresponding to adjacent zeros. But if the shock represents a local transition in an evolving wave form, the zeros depend on Z. They are the signal levels on either side of $\theta = \theta_s(Z)$, say, as determined along the characteristics $\mathrm{d}\theta/\mathrm{d}Z = q^2$ from the initial data $f(\theta)$. For a unit N-wave $f(\theta)$ the zeros corresponding to the tail shock are found (Lee-Bapty & Crighton 1986*a*) to be $f_1(Z) = 1 + O(Z-\frac{1}{2})$, $f_2 = 1 + O(Z-\frac{1}{2})^2$, $f_3 = -2 + O(Z-\frac{1}{2})$ near the range $Z = \frac{1}{2}$ at which the tail shock is formed (the tail shock initially present being immediately relieved by the cubic nonlinearity). The Taylor shock (after a range $|Z-\frac{1}{2}| = O(\epsilon^{\frac{1}{2}})$ in which the embryo-shock dynamics are governed by the ordinary quadratic Burgers' equation) provides a transition from f_1 to f_2. At $Z = Z_1 = 10$, however, f_2 and f_3 coalesce, and if $f_1(Z)$ and $f_2(Z)$ continue to be taken from the initial data as the levels on either side of the shock for $Z > Z_1$ then there is a singularity in the Taylor structure itself, because $f_1 > f_3 > f_2$ for $Z > Z_1$.

One might try to argue that some new mechanism should be introduced to resolve this singularity, or that the asymptotic analysis leading to this situation at Z_1 is incorrect. The first must be unnecessary, as the problem posed by (2.9) is well posed, and the second is denied by numerical results which confirm the final asymptotic scheme. What has happened at Z_1 is that a characteristic and the shock path have touched (see figure 1). If $f_1(Z)$ and $f_2(Z)$, as supplied by characteristics, are used to feed information to the shock for $Z > Z_1$, then the picture is as in figure 1 (*a*), and is clearly unacceptable, not only because of the singular shock structure, but also because of the crossing of characteristics away from the shock path.



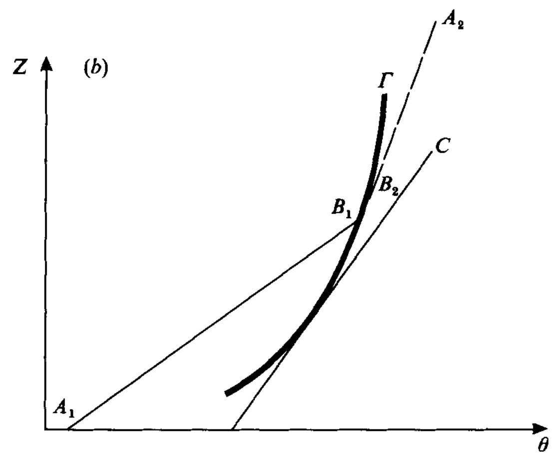


FIGURE 1. Shock path in (θ, Z)-plane for cubic modified Burgers' equation. (*a*) Γ is the shock path, C the tangential characteristic, touching Γ at (θ_1, Z_1), $Z_1 = 10$ for N-wave. Characteristics $A_1 B_1$, $A_2 B_2$ carry signals $f_1(Z), f_2(Z)$ to the shock, but $A_1 B_1$ crosses C away from the shock. Hence the picture shown is inadmissible for $Z > Z_1$. (*b*) Γ touches C at (θ_1, Z_1) and turns above C to prevent characteristic crossing as in (*a*). Characteristic $A_1 B_1$ carries signal $f_1(Z)$ to the shock at B_1. The signal at B_2 on the other side of the shock is given by $f_2 = -\frac{1}{2}f_1$, and propagates away from B_2 along the refracted characteristic $B_2 A_2$. No signal can reach the region between C and Γ and above (θ_1, Z_1) via characteristics from the initial data. All refracted characteristics such as $B_2 A_2$ are tangential to the shock path.

Two things are now necessary for an acceptable solution. First, $f_1 + 2f_2 \geqslant 0$ in order that $f_3 = -(f_1 + f_2)$ does not intrude between f_1 and f_2 and induce the shock singularity. Second, in order to avoid crossing of characteristics away from the shock, the shock must always turn to the left, as it were, above the tangent characteristic at any $Z \geqslant Z_1$, as in figure 1 (*b*). This can be shown to require $f_1 + 2f_2 \leqslant 0$. Consistency is achieved only if

$$f_1(Z) + 2f_2(Z) = 0, \quad Z \geqslant Z_1 \tag{4.1}$$

(where we assume $f_1 > f_2$).

Clearly, we must now abandon the idea that information can be passed in the usual (gasdynamic) way from the initial line via characteristics to both sides of the shock, with the shock having just a passive role. Figure 1 (*b*) shows that for $Z > Z_1$, information, in the form of $f_1(Z)$, comes along characteristics to the left of the shock, but that the shock now has an active role, *supplying* information to the lossless region to the right of the shock in the form of initial data $f_2(Z)$ for a new set of characteristics emanating from the shock. The shock amplitudes are related in the ratio $2:-1$ by the condition (4.1), which freezes coalescence of f_3 with f_2 for all $Z \geqslant Z_1$. Since the slopes of the characteristics for the cubic Burgers' equation are given by $\mathrm{d}\theta/\mathrm{d}Z = q^2$, the gradient in the (θ, Z)-plane increases by a factor 4 and the characteristic is refracted in crossing the shock. All such refracted characteristics do, in fact, *touch* the shock curve.

Eventually, for $Z > Z_2 = 90$ in the N-wave problem, the data refracted in this way through the tail shock become the source of information behind the head shock. No non-uniformity of significance is associated with this. The head shock remains described for finite Z by the Taylor form (2.11), but the head-shock amplitude a in (2.11) is now determined for $Z > Z_2$ by data propagated along characteristics refracted by the tail shock, rather than directly from the initial data. A much more complicated finite-Z representation applies in the case of an initial sinusoidal wave condition for the cubic Burgers' equation. Here there are repeated refractions as the characteristics pass through the periodic array of shocks. To find the lossless solution $q_0(\theta, Z)$ at any finite $Z > Z_1 = 9.601$ one has to trace the characteristic through (θ, Z) back to the line $Z = 0$, with gradient change by the factor 4 and signal change by the factor (-2) each time one of the shock paths is crossed. A recursive means of doing this can be devised (Lee-Bapty & Crighton 1986*a*), and the results can be favourably compared with asymptotic estimates which show, for example, that

$$f_1(Z) \sim 2\left(\frac{\pi}{3-2\ln 2}\right)^{\frac{1}{2}} Z^{-\frac{1}{2}} \quad \text{as } Z \to \infty.$$

We emphasize that all this relates to non-uniformities at finite Z which have to be resolved before large-Z non-uniformities can be addressed; also that only plane waves have so far been considered, for cubic nonlinearity. Wave-front area changes would not be expected to remove the difficulties just discussed, but higher-order nonlinearity (though perhaps not physically interesting at the moment) might lead to other non-uniformities arising from further or multiple coalescence of zeros of an nth order polynomial.

For $Z \geqslant Z_1$, with (4.1) imposed, there is exponential matching of the Taylor shock to the lossless solution on the left of the shock, algebraic on the right. (By Taylor shock here we mean (2.10) with zeros f_1, f_2, f_2 on the right.) It is therefore not easy to see what the signal level just to the right of the shock is, but it is fair to claim that figures 11 and 12 of Lee-Bapty & Crighton (1986*a*) do indeed confirm, from a finite-difference solution of (2.9), that condition (4.1) does lock the shock signals in the $2:-1$ ratio for $Z \geqslant Z_1$, which is the critical issue that distinguishes this study from that of quadratic nonlinearity. Large-Z non-uniformities are less interesting; they involve shock thickening, shock displacement and Taylor non-uniformity, much as for the generalized Burgers' equations of §3.

5. Other approaches

The difficulties for both analytical and numerical attacks on generalized Burgers' equations like (2.1) have been recognized for a long time, and numerous ingenious *ad hoc* methods devised to bridge the gap between the weak-shock-theory stage and the old-age stage (if it exists). These *ad hoc* methods (which often give remarkable correspondence of functional form with results from the matched-expansion scheme) have been discussed in detail in Nimmo & Crighton (1986, §10) and need no further comment here. It is, however, possible to make useful comparison of the predictions of the asymptotic scheme with numerical results.

Consider first cylindrical N-waves. These were first studied numerically by Sachdev & Seebass (1973) with a finite-difference scheme tested first against the exact solution for plane N-waves. The study concentrated on the 'lobe Reynolds number', which in our notation would be

$$R(Z) = \int_0^\infty q(\theta, Z) \frac{\mathrm{d}\theta}{\epsilon} \left(\frac{x_0}{x}\right)^{\frac{1}{2}}. \tag{5.1}$$

If (2.5) is integrated from 0 to ∞, and if it is assumed that the lossless solution for a unit N-wave can be used to give $\partial q/\partial\theta$ at $\theta = 0$ as $-1(Z+1)$, then one finds

$$R(Z) = \left(\frac{x_0}{x}\right)^{\frac{1}{2}} \left\{-\frac{1}{2\epsilon} + \int_0^Z \frac{G(Z)\,\mathrm{d}Z}{(Z+1)}\right\}. \tag{5.2}$$

Sachdev & Seebass compared an expression of essentially this form (with some numerical adjustment to make R agree with their smoothed initial N-wave) with computed results and found excellent agreement at ranges up to 500 (in our units). Over these ranges $|R|$ had changed by as much as a factor of 50. However, the value of ϵ chosen was 5×10^{-3}, so that the largest ranges considered were only of the order of the range $Z = \epsilon^{-1}$ at which the global non-uniformity for cylindrical N-waves is predicted, and a significant effect on $\partial q/\partial\theta$ at $\theta = 0$ would only be expected at still greater ranges. At those long ranges, where there is old-age decay, Crighton & Scott (1979, equation (3.36)) predict

$$q \sim C\left(\frac{-\theta}{\epsilon^2 Z^3}\right) \exp\left(\frac{\theta^2}{\epsilon Z^2}\right), \tag{5.3}$$

for $\theta = O(1)$ and $Z \gg \epsilon^{-1}$, with C an $O(1)$ positive constant. The ratio of $\partial q/\partial\theta$ at $\theta = 0$ from (5.3) to its value $-1/(Z+1)$ in the lossless solution is $O(\epsilon^{-2}Z^{-2})$, small in old age, and making (5.2) invalid there.

Sachdev & Seebass also computed wave forms at various stages of the evolution. These show the expected spreading of the whole wave and thickening of the shock, and just encompass the beginnings of old age at the largest ranges, but there is insufficient detail to permit a comparison with (5.3).

In recent work, Sachdev, Tikekar & Nair (1986) have greatly extended this work, using a pseudo-spectral approach in the initial stages where the waves are steep, and a finite-difference method later on. For cylindrical N-waves in the old-age phase they used a much larger value of ϵ, $\epsilon = 0.017$, for which the global non-uniformity is to be expected around $Z = 59$ (requiring, even so, the inversion of a 2500×2500 matrix). They found that old age was attained at about $Z = 99$, and that there the motion was very well described by (5.3) with $C = 0.34$ (giving a maximum amplitude of no more than 10^{-3} of the initial amplitude). Leibovich & Seebass (1974, p. 122) argued that the (θ, Z) dependence in old age should be as given in (5.3). However, they were unable to determine the ϵ-dependence, as their result involves as a factor the Reynolds number, whose dependence on ϵ for large Z cannot be found by arguments involving the lossless solution. It is, accordingly, an achievement of the matched-expansion approach that it is able to pin down everything in (5.3) save the numerical constant. Of course, final confirmation of the non-dependence of C on ϵ will only come when the calculations of Sachdev *et al.* (1986) have been repeated for a range of values of ϵ.

Nonetheless, the agreement of numerical results with the (θ, Z)-dependence of (5.3) is encouraging, and in contrast to work by Enflo (1981). He found a solution of the old-age linear equation of the form

$$q_0'(\theta', Z') = (Z')^{-2\alpha}\{A_E\,\Phi(\alpha, \tfrac{1}{2}; -y) + C_E\, y^{\frac{1}{2}}\Phi(\alpha + \tfrac{1}{2}, \tfrac{3}{2}; -y)\}, \tag{5.4}$$

where the scalings are

$$Z' = \epsilon Z, \quad \theta' = \epsilon^{\frac{1}{2}}\theta, \quad q(\theta', Z', \epsilon) \sim \epsilon^{\frac{1}{2}} q_0'(\theta', Z') + \dots, \tag{5.5}$$

y is written for θ'^2/Z'^2 and Φ is the confluent hypergeometric function normally written ${}_1F_1$. Symmetry demands $A_E = 0$. The form (5.3) of Crighton & Scott (1979) corresponds to $\alpha = 1$, which Enflo claims is not consistent with matching to the Taylor shock, preferring instead $\alpha = \frac{1}{2}$. He then attempts to construct a solution to the full cylindrical Burgers' equation, which in the present variables reads

$$\frac{\partial q_0'}{\partial Z'} - q_0' \frac{\partial q_0'}{\partial \theta'} = \tfrac{1}{2} Z' \frac{\partial^2 q_0'}{\partial \theta'^2}, \tag{5.6}$$

with an expansion

$$q_0' = C_E \frac{y^{\frac{1}{2}}}{Z'} \left\{ \Phi(1, \tfrac{3}{2}; -y) + \sum_{n=1}^{\infty} \frac{f_n(y)}{Z'^n} \right\} \tag{5.7}$$

starting with the old-age solution. The $f_n(y)$ are determined recursively, and from their asymptotics for large y it is claimed that the old-age form holds wherever $yZ' = O(\epsilon^{-1})$ and that this allows matching to the Taylor shock, giving

$$C_E = 1 - \tanh(\tfrac{1}{4}). \tag{5.8}$$

This procedure was effectively carried out by Fay (1931) for plane waves; he was able to sum the series explicitly and obtain an exact solution (the Fay solution) to the plane Burgers' equation, and from that, matching to the shock would indeed determine the old-age coefficient. No such summation was carried out by Enflo, however, and there is no possibility of matching a solution valid at large Z to one valid for $Z = O(1)$ without obtaining a more appropriate representation for $Z = O(\epsilon^{-1})$ than is given by (5.7). Thus we are unable to agree with Enflo, and point out that numerical work firmly favours (5.3) rather than (5.7). Similar reservations apply to the application of these ideas to sinusoidal cylindrical waves and cylindrical N-waves from a supersonic projectile (Enflo 1985*a*, *b*).

With regard to the latter problem, however, there is an important point to be made. In the absence of stratification, the Taylor description of the shock profile fails over 'ranges' $Z = O(\epsilon^{-1})$ for a thermoviscous fluid, and for a more realistic model of the atmosphere incorporating molecular relaxation effects as the principal dissipative mechanism and also density stratification there is presumably a corresponding range. If this argument is relevant to sonic booms from supersonic aircraft, it will be inappropriate to try to understand the rather large thickness (some tens of metres, at least one thousand times the Taylor thickness) in terms of fully developed shock structure. Explanation of these large thicknesses has long been a matter of controversy.

As a final comment on cylindrical N-waves, Sachdev *et al.* (1986) calculated the shock centre and compared it with the prediction of Crighton & Scott (1979, equation 3.24*b*) for the shock displacement due to diffusivity. Good agreement was found out to about $Z = 30$, again comparable with the range ϵ^{-1} ($\epsilon = 0.017$ here) for global violation of (A), (B) and (C) of §3. Good agreement was also found between numerical results for spherical N-waves and the displacement prediction of Crighton & Scott (1979, equation 3.24*c*). It is difficult to test the predictions of §3 for the non-uniformity leading to an evolutionary shock for spherical waves, and probably the best test is the old-age prediction

$$q \sim \frac{1}{6\pi^{\frac{1}{2}}} Z^{\frac{1}{2}}_{2} \frac{(-\theta)}{(\epsilon Z_0\, e^{Z/Z_0})^{\frac{3}{2}}} \exp\left\{ -\frac{\theta^2}{4\epsilon Z_0\, e^{Z/Z_0}} \right\}, \tag{5.9}$$

where $Z_2(\epsilon)$ is defined by

$$\frac{Z_2(\epsilon)}{\epsilon} = \exp\left(\frac{Z_2(\epsilon)}{Z_0}\right),$$

and marks the onset of old age. For the $\epsilon = 0.00431$ used by Sachdev *et al.* (1986) for spherical waves they find $Z_2 = 9.2$. The global non-uniformity leading to old age involves the spreading of the evolutionary shock to a thickness comparable with the overall N-wave scale. This scale at range Z is $Z^{\frac{1}{2}}$ (weak-shock theory places the shocks of a unit N-wave at $\pm(Z+1)^{\frac{1}{2}}$) and the predicted thickness of the error-function shock at Z_2 is $\epsilon^{\frac{1}{2}} \exp(Z_2/2Z_0) = 3.03$, so that the two are indeed comparable.

Sachdev *et al.* find that old age ensues at about $Z = 11$, and that thereafter the functional form of (5.9) agrees with numerical results over the whole wave very closely (to six decimal places!). The amplitude is not quite right, however; a factor 0.67 is needed on the right of (5.9) to make the results coincide. We have checked (5.9) again and are unable to offer any reason for this discrepancy.

The overall features of the asymptotic scheme are thus quite well borne out by the available numerical results. These are, however, really insufficient, and further studies covering a range of values of ϵ and Z_0 are definitely needed. Numerical studies for a different form of wave-front area variation are also needed; the diverging exponential horn is an obvious example, as it leads to a quite different type of long-range behaviour.

6. Conclusions

In this paper we have tried to describe the qualitative behaviour of nonlinear waves subject to geometrical wave-front area variations and to thermoviscous damping as they propagate over large ranges – and also the behaviour of plane waves (generally transverse, but possibly also longitudinal, as in Freon-13) with cubic or higher-order nonlinearity. Generalized or modified Burgers' equations describe these waves, respectively, and they have a very rich asymptotic structure in the small damping limit considered here. This structure is even richer if one takes a particular geometry and initial condition (e.g. spherical waves of initially sinusoidal form) and examines all the possibilities in an (ϵ, Z_0)-plane (Scott 1981*b*) instead of taking $Z_0 = O(1)$ or $\epsilon Z_0 = O(1)$ as was implicitly done above. In view of the proven absence of exact solutions by any of the methods developed in nonlinear wave theory (principally, but not exclusively, for dispersive rather than dissipative waves) a combination of asymptotic and numerical attacks is the only way forward. We believe that the asymptotic descriptions reported above – and in analytical detail in the papers cited – contain significant and interesting physics and are an indispensible *precursor* to numerical work, which so far seems adequately to confirm the asymptotics.

Much further numerical work is really needed, as emphasized in §5. One can also expect to find new scenarios for wave evolution not revealed so far. For example, if the initial condition gives a single one-signed pulse, then for plane waves the Reynolds number remains constant and the wave remains inherently nonlinear. How such a wave behaves under increasing or decreasing wave-front area variations is interesting and will be reported elsewhere, as will the combined effects of relaxation and stratification on the cylindrical N-waves of the sonic-boom problem. At the heart of all such studies, of course, is the Taylor shock structure, or its analogue, in local form, and the central issue is whether that structure can persist indefinitely, or whether its balance is upset by rapid changes outside the shock.

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